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CONSTRUCTION OF CLASSES OF EXPERIMENTAL DESIGNS USING TRANSVERSALS IN LATIN SQUARES AND HEDAYAT'S SUM COMPOSITION METHOD

by

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ABSTRACT

An experiment should be designed to satisfy the experimental considerations and objectives of the experiment. The experiment should not be designed to fit into known experimental designs if this results in a change in the desired findings. Consequently,new experimental designs and concepts need to be developed to satisfy new experimental requirements which continue to arise as a result of new investigations and research interests.

Several new types of experimental designs and some new methods of constructing known designs are considered in this paper. The ideas of parallel transversals in a latin square of order n and of common-parallel transversals in a pair of orthogonal latin squares of order n proved useful in constructing new as well as some previously known designs. The ideas and the procedure involved in the sum composition method of constructing a pair of orthogonal latin squares of order n proved very useful in the present work. The sum composition method is a new procedure for constructing a pair of orthogonal latin squares and was discovered by A. Hedayat. It derives its name from the fact that use is made of pairs of orthogonal latin squares of orders n_1 and n_2, such that $n_1 \div n_2 = n$ and $n_1 \geq 2n_2$, to produce the larger pair of orthogonal latin squares of order n. The method makes use of the projection of parallel and common-parallel transversals into the last n_2 rows and columns of the square of order n.

It is shown how to construct Youden square and balanced incomplete block designs using the sum composition method. Plans of partially balanced latin rectangle designs, T: TT type designs, supplemented balanced and partially balanced designs, augmented designs, and designs for successive experiments conducted on the same experimental units were constructed using the ideas and procedures in the sum composition method. Most of the plans constructed will have a relatively simple statistical analysis. The analyses are not given in the present paper.

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1. Introduction

A new method of constructing a pair of mutually orthogonal latin squares of order n has been discovered by A. Hedayat (see Keller [1969], Federer et al. [1969], and Hedayat and Seiden [1970]). The method is called the sum composition method since it makes uses of pairs of mutually orthogonal latin squares of orders n_1 and n_2, such that $n_1 + n_2 = n$, to produce the larger pair of mutually orthogonal latin squares of order n. The method makes use of parallel and common-parallel transversals in latin squares of order n_1, for $n_1 > n_2$, and of the projection of the transversals into the last n_2 rows and n_2 columns of a square of side n.

For the sum composition method the k parallel transversals may be selected in many ways but we shall restrict selection to be among the k parallel transversals which when projected produce a balanced incomplete block design arrangement. Within this restriction, it is shown how to construct plans for balanced incomplete block designs, Youden square designs, partially balanced latin rectangle designs, T:TT designs, P:PP designs, supplemented balanced and partially balanced designs of the O:SS type and O:S_pS_p type, augmented designs of various types, and several types of designs for successive (or simultaneous) experiments on the same set of experimental material. Illustrative examples of designs in each of the classes are presented for each of the classes of designs.

The statistical analysis for most of the designs is relatively simple computationally. The analyses are not presented in the present paper. Also, the randomization procedure for the various classes is not presented. However,

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for a single experiment a permutation of the rows, columns, and treatments in any given plan should suffice for most purposes.

2. Definition and Notation

In the following, the symbol $O(n,t)$ denotes a set of t mutually orthogonal latin squares of order n. A transversal, or directrix, of a latin square $L(n)$ of order n on an n-set Σ is a collection of n cells such that the entries of these cells exhaust the set Σ and every row and every column of $L(n)$ is represented in this collection. Two transversals in $L(n)$ are said to be parallel if they have no cell in common. A collection of n cells is said to form a common transversal for an $O(n,t)$ set if the collection is a transversal for each of the t latin squares. Similarly, two common transversals are said to be common-parallel transversals if they have no cell of the squares in common. In the following $O(4,2)$ set, two common-parallel transversals are presented. One of the common transversals is indicated by underlining the elements in the two squares and the other common transversal is indicated by the parentheses around the elements in the two squares.

1	2	(3)	$\underline{4}$
(2)	$\underline{1}$	4	3
$\underline{3}$	(4)	1	2
4	3	$\underline{2}$	(1)

1	2	(3)	$\underline{4}$
(4)	$\underline{3}$	2	1
$\underline{2}$	(1)	4	3
3	4	$\underline{1}$	(2)

Not all latin squares of order n have a transversal, and still fewer latin squares of order n have n transversals. For any given latin square of order n, it may be extremely difficult to determine whether or not one or more transversals exist even though a relatively simple procedure has been developed for constructing a latin square with a transversal by Hedayat and Federer [1970]. However, if an O(n,2) set of latin squares is available, it is a simple matter to find the n transversals in each of the two squares. One simply superimposes one square on top of the other and notes the n positions for a single element of one of the two squares. These n positions form a transversal for the second square. Continuing this process for all n elements in the first square, the n transversals of the second square are obtained. The n transversals of the first square may be obtained similarly. This method of finding a square with n transversals is available for all n except n = 2 and 6. O(n,2) sets for all odd n are easily obtained by a cyclical permutation of the elements in the set (see Hedayat and Federer [1969]). A new method of constructing an O(n,2) set has been developed by A. Hedayat (see Keller [1969], Federer *et al.* [1969], Hedayat and Seiden [1970]); this method has been designated as the *sum composition method* since a latin square of order $n = n_1 \div n_2$ is produced from two latin squares of orders n_1 and n_2. The method is especially useful for even n and for $n = 4q \div 2$ where $q = 2,3,\ldots$. The group construction method (see Federer *et al.* [1969]) are useful for constructing O(n,2) sets when n is not of the form $4q + 2$. Any O(n,2) set suffices to determine the n parallel transversals in the two squares.

If an O(n,3) set exists, then it is a relatively simple matter to find the n common-parallel transversals for any pair of the latin squares in the set by using the third square in the set. One of the latin squares is superimposed upon each of the other two latin squares. The n positions of a specified element in the first square determine a common transversal in the other two squares. Continuing this process for the remaining n-1 elements of the first square, the n common-parallel transversals of the other two squares are obtained. To illustrate, consider the following O(4,3) set:

$L_1(4)$

1^*	$\underline{2}$	3^+	4^-
$\underline{2}$	1^*	4^-	3^+
3^+	4^-	1^*	$\underline{2}$
4^-	3^+	$\underline{2}$	1^*

$L_2(4)$

1^*	$\underline{2}$	3^+	4^-
$\underline{4}$	3^*	2^-	1^+
2^+	1^-	4^*	$\underline{3}$
3^-	4^+	$\underline{1}$	2^*

$L_3(4)$

1^*	$\underline{2}$	3^+	4^-
$\underline{3}$	4^*	1^-	2^+
4^+	3^-	2^*	$\underline{1}$
2^-	1^+	$\underline{4}$	3^*

The four common-parallel transversals in $L_2(4)$ and $L_3(4)$ are those indicated with the symbols *, +, -, _, which correspond respectively to elements 1, 3, 4, and 2 in $L_1(4)$.

An O(n,t) set suffices to obtain the n common-parallel transversals in t-1 of the latin squares of order n. The procedure for doing this is a straightforward extension of the procedure described for an O(n,3) set.

3. The Sum Composition Method of Constructing O(n,2) Sets

The method of sum composition of O(n,2) sets is described in detail by Hedayat and Seiden [1970]. They proved, among other results, that a sufficient condition for the construction of an O(n,2) set by the method of sum composition is that an $O(n_1,2)$ set and an $O(n_2,2)$ set be available, that $n_1 + n_2 = n$, that $n_1 \geq 2n_2$, that the $O(n_1,2)$ set contains at least $2n_2$ common-parallel transversals, and that either (i) $7 \leq n_1 = p^\alpha$, p an odd prime or $8 \leq n_1 = 2^\alpha$, α a positive integer and $n_2 = (n_1-1)/2$ or $n_2 = n_1/2$ respectively; (ii) $7 \leq n_1 = p^\alpha$, p a prime having any of the following forms: $3m + 1$, $8m + 1$, $8m + 3$, $24m + 11$, $60m + 23$, or $60m + 47$ and $n_2 = 3$; (iii) $n_1 = p^\alpha$, p a prime of the form $8m + 1$, or $8m + 3$, $m \neq 0$ and $n_2 = 4$.

An example of the method will suffice for the purposes of this paper. Let $n_1 = 7$ and $n_2 = 3$; let the O(7,2) and O(3,2) sets be the following:

$C_1(10)$

0	$\underline{1}$	2^*	3	4^+	5	6			
$\underline{2}$	3^*	4	5^+	6	0	1			
4^*	5	6^+	0	1	2	$\underline{3}$			
6	0^+	1	2	3	$\underline{4}$	5^*			
1^+	2	3	4	$\underline{5}$	6^*	0			
3	4	5	$\underline{6}$	0^*	1	2^+			
5	6	$\underline{0}$	1^*	2	3^+	4			
							7	8	9
							8	9	7
							9	7	8

$C_2(10)$

0	1	2	(3	4	5)	6^-			
4	5	(6	0	1)	2^-	3			
1	(2	3	4)	5^-	6	0			
(5	6	0)	1^-	2	3	4			
2	3)	4^-	5	6	0	(1			
6)	0^-	1	2	3	(4	5			
3^-	4	5	6	(0	1	2)			
							7	8	9
							9	7	8
							8	9	7

To demonstrate that the above marked left diagonals are common-parallel transversals, we need to consider the $O(7,3)$ set of which the preceding two squares and the following square are members:

$L_1(7) =$

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

The left diagonal in $C_1(10)$ containing the elements with an underline corresponds to the left diagonal containing the element 1 in $L_1(7)$; the left diagonal in $C_1(10)$ marked with an asterisk corresponds to the left diagonal containing the element 2 in $L_1(7)$; the left diagonal in $C_1(10)$ marked with a plus sign superscript corresponds to the left diagonal containing the element 4 in $L_1(7)$; the left diagonal in $C_2(10)$ marked with a left parenthesis corresponds to the left diagonal containing the element 3 in $L_1(7)$; the left diagonals marked with a right parenthesis and with a minus superscript in $C_2(10)$ corresponds to those in $L_1(7)$ containing elements 5 and 6 respectively. The remaining common transversal is unmarked and it corresponds to the diagonal in $L_1(7)$ containing the element 0.

The next step after selecting the $n_2 = 3$ parallel transversals in each of the two latin squares of order $n_1 = 7$ in $C_1(n_1 + n_2) = C_1(10)$ and $C_2(n_1 + n_2) = C_2(10)$ is to project the elements from each selected transversal into an empty row and into an empty column of both $C_1(n_1 + n_2)$ and $C_2(n_1 + n_2)$. For $n_1 + n_2 = 10$, the following results:

$C_3(10)$

0			3		5	6	$\underline{1}$	2^*	4^+
		4		6	0	1	$\underline{2}$	3^*	5^+
	5		0	1	2		$\underline{3}$	4^*	6^+
6		1	2	3			$\underline{4}$	5^*	0^+
	2	3	4			0	$\underline{5}$	6^*	1^+
3	4	5			1		$\underline{6}$	0^*	2^+
5	6			2		4	$\underline{0}$	1^*	3^+
$\underline{2}$	$\underline{1}$	$\underline{0}$	$\underline{6}$	$\underline{5}$	$\underline{4}$	$\underline{3}$	7	8	9
4^*	3^*	2^*	1^*	0^*	6^*	5^*	8	9	7
1^+	0^+	6^+	5^+	4^+	3^+	2^+	9	7	8

$C_4(10)$

0	1	2		4			6^-	5)	(3
4	5		0			3	2^-	1)	(6
1		3			6	0	5^-	4)	(2
	6			2	3	4	1^-	0)	(5
2			5	6	0		4^-	3)	(1
		1	2	3		5	0^-	6)	(4
	4	5	6		1		3^-	2)	(0
3^-	0^-	4^-	1^-	5^-	2^-	6^-	7	8	9
6)	3)	0)	4)	1)	5)	2)	9	7	8
(5	(2	(6	(3	(0	(4	(1	8	9	7

The last step in constructing an O(10,2) set is to insert one element from the latin square of order $n_2 = 3$ in each of the $n_2 = 3$ transversals to obtain the following O(10,2) set:

$L_1(10)$

0	7	8	3	9	5	6	1	2	4
7	8	4	9	6	0	1	2	3	5
8	5	9	0	1	2	7	3	4	6
6	9	1	2	3	7	8	4	5	0
9	2	3	4	7	8	0	5	6	1
3	4	5	7	8	1	9	6	0	2
5	6	7	8	2	9	4	0	1	3
2	1	0	6	5	4	3	7	8	9
4	3	2	1	0	6	5	8	9	7
1	0	6	5	4	3	2	9	7	8

$L_2(10)$

0	1	2	7	4	8	9	6	5	3
4	5	7	0	8	9	3	2	1	6
1	7	3	8	9	6	0	5	4	2
7	6	8	9	2	3	4	1	0	5
2	8	9	5	6	0	7	4	3	1
8	9	1	2	3	7	5	0	6	4
9	4	5	6	7	1	8	3	2	0
3	0	4	1	5	2	6	7	8	9
6	3	0	4	1	5	2	9	7	8
5	2	6	3	0	4	1	8	9	7

Latin squares of order $n_1 = 11$ and $n_2 = 5$ may be used to construct an $O(16,2)$ set by the method of sum composition. Let the following latin square of order 11 be used to determine the common parallel transversals in the construction of $C_1(16)$ and $C_2(16)$ in the same manner as for $n_1 = 7$:

$L_1(11) =$

0	1	2	3	4	5	6	7	8	9	10
10	0	1	2	3	4	5	6	7	8	9
9	10	0	1	2	3	4	5	6	7	8
8	9	10	0	1	2	3	4	5	6	7
7	8	9	10	0	1	2	3	4	5	6
6	7	8	9	10	0	1	2	3	4	5
5	6	7	8	9	10	0	1	2	3	4
4	5	6	7	8	9	10	0	1	2	3
3	4	5	6	7	8	9	10	0	1	2
2	3	4	5	6	7	8	9	10	0	1
1	2	3	4	5	6	7	8	9	10	0

$C_1(16)$

0	1	2	3	4	5	6	7	8	9	10					
1	2	3	4	5	6	7	8	9	10	0					
2	3	4	5	6	7	8	9	10	0	1					
3	4	5	6	7	8	9	10	0	1	2					
4	5	6	7	8	9	10	0	1	2	3					
5	6	7	8	9	10	0	1	2	3	4					
6	7	8	9	10	0	1	2	3	4	5					
7	8	9	10	0	1	2	3	4	5	6					
8	9	10	0	1	2	3	4	5	6	7					
9	10	0	1	2	3	4	5	6	7	8					
10	0	1	2	3	4	5	6	7	8	9					
											11	12	13	14	15
											12	13	14	15	11
											13	14	15	11	12
											14	15	11	12	13
											15	11	12	13	14

$C_2(16)$

0	1	2	3	4	5	6	7	8	9	10					
2	3	4	5	6	7	8	9	10	0	1					
4	5	6	7	8	9	10	0	1	2	3					
6	7	8	9	10	0	1	2	3	4	5					
8	9	10	0	1	2	3	4	5	6	7					
10	0	1	2	3	4	5	6	7	8	9					
1	2	3	4	5	6	7	8	9	10	0					
3	4	5	6	7	8	9	10	0	1	2					
5	6	7	8	9	10	0	1	2	3	4					
7	8	9	10	0	1	2	3	4	5	6					
9	10	0	1	2	3	4	5	6	7	8					
											11	12	13	14	15
											13	14	15	11	12
											15	11	12	13	14
											12	13	14	15	11
											14	15	11	12	13

The right diagonals from $C_1(16)$ to be projected into the last $n_2 = 5$ rows and $n_2 = 5$ columns are those corresponding to the right diagonals in $L_1(11)$ containing the elements of 0, 4, 7, 9, and 10. Here, as in the case with $C_1(10)$ and $C_2(10)$, diagonals were selected such that the projected elements into rows would form a balanced incomplete block design arrangement within columns, and the projected elements into columns would form a balanced incomplete block design arrangement within rows.

In selecting the particular transversals giving this arrangement, one need only check to ascertain that <u>one</u> element occurs equally frequent with all other elements. The nature of projection of transversals makes it unnecessary to check frequency of occurrence for all pairs of elements. Also, from any balanced incomplete block design plan for $v = n_1 = b$, $k = r$, and $\lambda = k(k-1)/(n_1-1)$, the elements in any single block may be used to denote the transversals to be projected; then, from a <u>different</u> balanced incomplete block plan, the elements of any block containing k of the remaining $n_1 - k$ symbols may be used to denote the transversals to be projected in the second square. Not all balanced incomplete block design plans can be utilized to determine the transversals to be projected in the 2nd square. Another permutation of the transversals and the columns (or rows) into which they are projected may be necessary in order to form an $O(n,2)$ set and to obtain the balanced incomplete block design arrangement in the last k rows and columns.

The parallel transversals selected for projection in $C_2(16)$ are those corresponding to right diagonals in $L_1(11)$ containing the elements 1, 2, 3, 5, and 8. The resulting $C_3(16)$ and $C_4(16)$ plans follow:

$C_3(16)$

	1	2	3		5	6		8			0	4	7	9	10
		3	4	5		7	8		10		2	6	9	0	1
			5	6	7		9	10		1	4	8	0	2	3
3				7	8	9		0	1		6	10	2	4	5
	5				9	10	0		2	3	8	1	4	6	7
5		7				0	1	2		4	10	3	6	8	9
6	7		9				2	3	4		1	5	8	10	0
	8	9		0				4	5	6	3	7	10	1	2
8		10	0		2				6	7	5	9	1	3	4
9	10		1	2		4				8	7	0	3	5	6
10	0	1		3	4		6				9	2	5	7	8
0	2	4	6	8	10	1	3	5	7	9	11	12	13	14	15
7	9	0	2	4	6	8	10	1	3	5	12	13	14	15	11
4	6	8	10	1	3	5	7	9	0	2	13	14	15	11	12
2	4	6	8	10	1	3	5	7	9	0	14	15	11	12	13
1	3	5	7	9	0	2	4	6	8	10	15	11	12	13	14

$C_4(16)$

0				4		6	7		9	10	8	5	3	2	1
2	3				7		9	10		1	0	8	6	5	4
4	5	6				10		1	2		3	0	9	8	7
	7	8	9				2		4	5	6	3	1	0	10
8		10	0	1				5		7	9	6	4	3	2
10	0		2	3	4				8		1	9	7	6	5
	2	3		5	6	7				0	4	1	10	9	8
3		5	6		8	9	10				7	4	2	1	0
	6		8	9		0	1	2			10	7	5	4	3
		9		0	1		3	4	5		2	10	8	7	6
			1		3	4		6	7	8	5	2	0	10	9
6	9	1	4	7	10	2	5	8	0	3	11	12	13	14	15
1	4	7	10	2	5	8	0	3	6	9	13	14	15	11	12
9	1	4	7	10	2	5	8	0	3	6	15	11	12	13	14
5	8	0	3	6	9	1	4	7	10	2	12	13	14	15	11
7	10	2	5	8	0	3	6	9	1	4	14	15	11	12	13

Inserting an element from the latin square of order $n_2 = 5$ into a transversal which was projected produces a latin square of order 16. The two squares resulting in an O(16,2) set are:

$L_1(16)$

11	1	2	3	12	5	6	13	8	14	15	0	4	7	9	10
15	11	3	4	5	12	7	8	13	10	14	2	6	9	0	1
14	15	11	5	6	7	12	9	10	13	1	4	8	0	2	3
3	14	15	11	7	8	9	12	0	1	13	6	10	2	4	5
13	5	14	15	11	9	10	0	12	2	3	8	1	4	6	7
5	13	7	14	15	11	0	1	2	12	4	10	3	6	8	9
6	7	13	9	14	15	11	2	3	4	12	1	5	8	10	0
12	8	9	13	0	14	15	11	4	5	6	3	7	10	1	2
8	12	10	0	13	2	14	15	11	6	7	5	9	1	3	4
9	10	12	1	2	13	4	14	15	11	8	7	0	3	5	6
10	0	1	12	3	4	13	6	14	15	11	9	2	5	7	8
0	2	4	6	8	10	1	3	5	7	9	11	12	13	14	15
7	9	0	2	4	6	8	10	1	3	5	12	13	14	15	11
4	6	8	10	1	3	5	7	9	0	2	13	14	15	11	12
2	4	6	8	10	1	3	5	7	9	0	14	15	11	12	13
1	3	5	7	9	0	2	4	6	8	10	15	11	12	13	14

$L_2(16)$

0	11	12	13	4	14	6	7	15	9	10	8	5	3	2	1
2	3	11	12	13	7	14	9	10	15	1	0	8	6	5	4
4	5	6	11	12	13	10	14	1	2	15	3	0	9	8	7
15	7	8	9	11	12	13	2	14	4	5	6	3	1	0	10
8	15	10	0	1	11	12	13	5	14	7	9	6	4	3	2
10	0	15	2	3	4	11	12	13	8	14	1	9	7	6	5
14	2	3	15	5	6	7	11	12	13	0	4	1	10	9	8
3	14	5	6	15	8	9	10	11	12	13	7	4	2	1	0
13	6	14	8	9	15	0	1	2	11	12	10	7	5	4	3
12	13	9	14	0	1	15	3	4	5	11	2	10	8	7	6
11	12	13	1	14	3	4	15	6	7	8	5	2	0	10	9
6	9	1	4	7	10	2	5	8	0	3	11	12	13	14	15
1	4	7	10	2	5	8	0	3	6	9	13	14	15	11	12
9	1	4	7	10	2	5	8	0	3	6	15	11	12	13	14
5	8	0	3	6	9	1	4	7	10	2	12	13	14	15	11
7	10	2	5	8	0	3	6	9	1	4	14	15	11	12	13

This discussion will not be carried further here since the above is sufficient to illustrate a number of types of experimental design plans that may be constructed using the above procedures and concepts.

4. Youden Square and Balanced Incomplete Block Design Plans Constructed from Common-Parallel Transversals in an $O(n_1 + n_2, 2)$ Set

In a balanced incomplete block design with v treatments (or elements) replicated r times in the v = n blocks of size r = k, every pair of treatments occurs together $\lambda = r(k-1)/(v-1) = k(k-1)/(n-1)$ times, for $\lambda = 1,2,\ldots$. A Youden square design has balanced incomplete block design properties in columns and has randomized complete block properties in rows. (The randomized complete block property in rows means the treatment parameters are orthogonal to the row parameters.) If T denotes the (totally) balanced property of one set of parameters within a second set, if 0 denotes orthogonality of two sets of parameters, and if rows, columns, and treatments are always referred to in this order, then a latin square design is of the 0:00 type, and a Youden square design is of an 0:0T type. In the latter type, the 0 to the left of the colon denotes the fact that columns are orthogonal to rows; the 0T designation to the right of the colon denotes the fact that treatments are orthogonal to rows and totally balanced with respect to columns. This symbolism was introduced and used to designate various types of designs for successive experiments by Hoblyn, Pearce, and Freeman[(1954], by Pearce [1960, 1963], by Clarke [1963], and by Freeman [1964].

In the second step of the method of sum composition of a pair of orthogonal latin squares of side $n = n_1 + n_2$, k transversals were projected into the last k rows and the last k columns of $C_1(n_1 + n_2)$ and $C_2(n_1 + n_2)$ to form $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ plans. When $\lambda = k(k-1)/(n-1)$ = an integer, a Youden square plan results from an appropriate selection of the k **transversals**. The following Youden square plans were obtained from the projected sets in $C_3(10)$ and $C_4(10)$:

From rows 8, 9, and 10 of $C_3(10)$

2	1	0	6	5	4	3
4	3	2	1	0	6	5
1	0	6	5	4	3	2

From columns 8, 9, and 10 of $C_3(10)$

1	2	3	4	5	6	0
2	3	4	5	6	0	1
4	5	6	0	1	2	3

From rows 8, 9, and 10 of $C_4(10)$

3	0	4	1	5	2	6
6	3	0	4	1	5	2
5	2	6	3	0	4	1

From columns 8, 9, and 10 of $C_4(10)$

6	2	5	1	4	0	3
5	1	4	0	3	6	2
3	6	2	5	1	4	0

In each of the above four plans the elements (or treatments) are in a balanced incomplete block design arrangement within columns with $\lambda = 1$ and $k = r = 3$. The row and element parameters are orthogonal to each other.

If the remaining four transversals in $C_3(10)$ and $C_4(10)$ are projected into four rows and four columns, four more Youden square plans result. The parameters of these plans are $v = n_1 = 7$, $k = r = 4$, and $\lambda = k(k-1)/(n-1) = 2$.

Let us now turn to plans $C_3(16)$ and $C_4(16)$. The last five rows and first eleven columns and the first eleven rows and the last five columns form four different Youden square plans with the parameters $v = n = 11$, $r = k = 5$, and $\lambda = k(k-1)/(n-1) = 2$. Projection of the remaining six transversals in the first eleven rows and eleven columns of $C_3(16)$ and $C_4(16)$ into rows and columns produces a quartet of Youden square plans with the parameters $v = b = 11$, $k = r = 6$, and $\lambda = 3$.

It should be noted here that not all $O(n,2)$ sets constructed by the sum composition method and the projection of $k = n_2$ transversals into the last k rows and the last k columns of $C_1(n_1 + n_2)$ and $C_2(n_1 + n_2)$ produce a Youden square plan. It is necessary to find k transversals which produce blocks for which <u>one</u> of the elements occurs λ times with every other element. It is not necessary to check occurrences for all pairs of elements because of the nature of parallel transversals. Then, from among the remaining $n-k = n_1$ transversals, it is necessary to find k different transversals such that the projected transversals into the last k rows and last k columns produce a Youden square plan.

The following is a list of values of $n_1 = v \leq 50$ and $k = r < n_1-1$ for which $\lambda = k(k-1)/(n_1-1)$ is an integer, where v is the number of treatments or elements in the set and k is block size:

$v = n_1$	$k = r$	λ	$v = n_1$	$k = r$	λ	$v = n_1$	$k = r$	λ
7	3	1	27	13	6	39	19	9
7	4	2	27	14	7	39	20	10
11	5	2	29	8	2	41	16	6
11	6	3	29	21	15	41	25	15
13	4	1	31	6	1	43	7	1
13	9	6	31	25	20	43	36	30
15	7	3	31	10	3	43	15	5
15	8	4	31	21	14	43	28	18
16	6	2	31	15	7	43	21	10
16	10	6	31	16	8	43	22	11
19	9	4	34	12	4	45	12	3
19	10	5	34	22	14	45	33	24
21	5	1	35	17	8	46	10	2
21	16	12	35	18	9	46	36	28
23	11	5	36	15	6	47	23	11
23	12	6	36	21	12	47	24	12
25	9	3	37	9	2	49	16	5
25	16	10	37	28	21	49	33	22

Youden square plans may also be formed by deleting one row (or column) from any latin square plan. This procedure does not involve the use of projection of parallel transversals. These plans are not listed above.

The above table has some interesting features in connection with the sum composition method. So far, this method has not produced an $O(n,3)$ set. The numbers $n_1 = 31$ and 43 appear to be interesting candidates to produce an $O(31 + 5, 3)$ set and an $O(43 + 7, 3)$ set. The parallel transversals which when projected should form rows of different Youden squares. That is, to produce an $O(36,3)$ set, take five transversals of the six which produce a 6-row $\times$ 31-column Youden square, five from the ten which form a 10-row $\times$ 31-column Youden square, and five from the 15 which form a 15-row $\times$ 31-column Youden square. Perhaps an $O(36,5)$ set could be formed in this manner. Also, it is interesting to note that whenever n_1 is of the form $4q + 3$, $q=1,2,\cdots$, Youden square plans are formed for $k = (n_1-1)/2$ and $k = (n_1+1)/2$. The corresponding λ values are $\lambda = (n_1-3)/4 = (k-1)/2$ and $\lambda = (n_1+1)/4 = k/2$. The numbers 31 and 43 are of the form $4q + 3$, but additional partitions producing Youden square plans are available for these numbers. Extension of the above table to values of 100 or higher may throw additional light on these features.

5. T:TT Type Plans for n_1 Rows, Columns, and Treatments

In the upper left 7×7 partitioning of $C_3(10)$ and $C_4(10)$ the treatments form a balanced incomplete block design arrangement within rows and within columns with $\lambda = 2$. With the $21 = kn_1$ entries omitted from the k transversals from this part of $C_3(10)$ and $C_4(10)$, the columns are in a balanced arrangement within rows, and *vice versa*. In this design $\lambda = 2$ for columns within rows and for treatments within rows and within columns.

Likewise, in the upper left 11×11 partitioning of $C_3(16)$ and $C_4(16)$ with 6(11) entries remaining, the treatments form a balanced incomplete block design arrangement within rows and within columns with $\lambda = 3$; the columns are in a balanced incomplete block design arrangement within rows, and *vice versa*, with $\lambda = 3$ also.

Both of the above designs are of the T:TT type. This appears to be a new class of designs for situations where the kn_1 deleted entries may be left blank, where the kn_1 deleted entries may be replaced by a dummy treatment, or where additional treatments are to be inserted in the kn_1 spaces. This last class of plans will be considered later. Thus, from the upper left $n_1 \times n_1$ part of the $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ plans, a class of designs of the T:TT type may be easily constructed from one of the steps in the sum composition method of constructing an $O(n_1 + n_2, 2)$ set. Since the block size and the value of λ for treatments are identical within rows and within columns, and since a balanced arrangement is obtained, this results in a relatively simple statistical analysis.

6. Partially Balanced Latin Rectangle and P:PP Plans

Instead of projecting the k transversals which result in a Youden square plan, fewer than the k transversals resulting in a balanced incomplete block design arrangement may be projected to produce a partially balanced latin rectangle arrangement. For example, in $C_3(10)$ and $C_4(10)$ the use of only two of the three selected projected diagonals results in a 2-row $\times$ 7-column latin rectangle; the set of three projected transversals considered are those resulting in a Youden square plan. In the $C_3(16)$ and $C_4(16)$ plans, two, three, or four of the five selected transversals producing a Youden square plan may be used to produce a partially balanced 2-row, 3-row, or 4-row by 11-column latin rectangle. Since the selection of fewer than k transversals is from the k transversals producing a balanced arrangement, the resulting latin rectangles will be as nearly balanced as possible.

If the k parallel transversals producing a Youden square plan are projected and if 1,2,... additional parallel transversals are projected, the resulting plans will be in the class of partially balanced latin rectangle plans, except in those instances where sufficient parallel transversals are projected to result in a balanced arrangement. In $C_3(10)$ and $C_4(10)$, the projection of any pair of parallel transversals from among the remaining four results in a 5-row × 7-column partially balanced latin rectangle plan; the projection of any triplet of parallel transversals from among the remaining four results in a 6-row × 7-column Youden square plan. In $C_3(16)$ and $C_4(16)$, the projection of an additional one, two, three, or four additional parallel transversals from among the remaining six results in a partially balanced latin rectangle. The projection of any five additional parallel transversals to the five already projected results in a Youden square plan.

In the $n_1 \times n_1$ part of plans $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ which remains after the projection of fewer than (or more than) the k transversals resulting in a balanced arrangement, the elements or the treatments in the resulting plan have the partially balanced incomplete block design property within rows and within columns. Likewise, the columns have the partially balanced incomplete block design property within rows, and <u>vice versa</u>. Thus, the remaining entries in the $n_1 \times n_1$ part, after projecting k-q, q = 1,2,...,k-2, parallel transversals, form a P:PP type design; P denotes partially balanced.

Likewise, after the projection of k + 1, k + 2,...,n_1-2 parallel transversals where the first k form a Youden square plan, the remaining entries in the $n_1 \times n_1$ parts of $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ plans form a P:PP type plan. The P:PP class of designs is useful in the same situations discussed for the T:TT class of designs.

7. Supplemented Balanced and Partially Balanced Plans

A supplemented balanced plan is one in which one element or treatment appears once in each of the n_1 rows and once within each of the n_1 columns and n_1 elements or treatments appear n_1-1 times in the n_1 rows and in the n_1 columns (see Pearce [1960]). The treatment appearing n_1 times is inserted in a transversal of a latin square of order n_1; the treatments which appear n_1-1 times form a balanced arrangement within rows and within columns. This type of arrangement of the $n_1 + 1$ elements or treatments in the n_1-row × n_1-column arrangement is denoted as supplemented balance and is denoted by the symbol S. Since rows and columns are orthogonal to each other, the above plan is denoted as an O: SS type of plan.

We may extend this idea by inserting k additional treatments in the places occupied by the projected k parallel transversals such that one additional treatment is placed on one of the projected parallel transversals in the $n_1 \times n_1$ part of $C_3(n_1+n_2)$ and $C_4(n_1+n_2)$ forming the T: TT plan. This results in $n_1 + k$ elements or treatments in an $n_1 \times n_1$ square with the k additional treatments occurring n_1 times and the original n_1 treatments occurring n_1-k times. Each of the k additional treatments are in a randomized block arrangement among themselves and are in a supplemented balanced arrangement to the n_1 original treatments in the T: TT plan. This class of plans for $n_1 + k$ treatments in an $n_1 \times n_1$ square is denoted as an O: SS plan. (This plan could also be denoted as an augmented T: TT plan to conform with notation used for other augmented plans presented later.)

To illustrate, let the k = 3 additional treatments be denoted as A, B, and C and use the T: TT plan from $C_3(10)$ to obtain the following plan:

0	A	B	3	C	5	6
A	B	4	C	6	0	1
B	5	C	0	1	2	A
6	C	1	2	3	A	B
C	2	3	4	A	B	0
3	4	5	A	B	1	C
5	6	A	B	2	C	4

A, B, and C each occur $n_1 = 7$ times and 0, 1, 2, 3, 4, 5, and 6 each occur $n_1 - k = 4$ times. If four appropriate transversals are projected (e.g., those remaining in $C_3(10)$), the resulting design is of the T:TT type with 49-4(7) = 21 entries; four additional treatments, A, B, C, and D, could be added to obtain a supplemented balanced plan with 7 + 4 = 11 treatments. For example, in $C_1(10)$, project transversals corresponding to elements 0, 3, 5, and 6 of $L_1(7)$ and replace these transversals with elements A, B, C, and D, respectively, to obtain:

A	1	2	B	4	C	D
2	3	B	5	C	D	A
4	B	6	C	D	A	3
B	0	C	D	A	4	5
1	C	D	A	5	6	B
C	D	A	6	0	B	2
D	A	0	1	B	3	C

In the above plan the T:TT plan with $n_1 = 7 = v$, $r = k = 3$, and $\lambda = 1$ was augmented with four additional treatments each occurring $n_1 = 7$ times. Alternatively, one could start with a T:00 plan as follows,

A			B		C	D
		B		C	D	A
	B		C	D	A	
B		C	D	A		
	C	D	A			B
C	D	A			B	
D	A			B		C

and augment this plan with treatments 0, 1, 2, 3, 4, 5, and 6 in such a manner that these augmented treatments form a balanced incomplete block design arrangement within rows and within columns. The preceding utilizes the fact that the k projected transversals produce a balanced incomplete block design arrangement.

If plans of the P:PP type were augmented with additional treatments each occurring n_1 times, the resulting plan would be denoted as one with supplemented partial balance. Denoting this by S_p, an $0{:}S_pS_p$ plan for $n_1 = 7$ treatments each occurring five times and two additional treatments, A and B, each occurring $n_1 = 7$ times may be obtained by removing two transversals, say transversals corresponding to elements 0 and 6 of $L_1(7)$, from $C_1(10)$ and replacing these with A and B as follows:

A	1	2	3	4	5	B
2	3	4	5	6	B	A
4	5	6	0	B	A	3
6	0	1	B	A	4	5
1	2	B	A	5	6	0
3	B	A	6	0	1	2
B	A	0	1	2	3	4

Also, one could start with a plan of the type,

A	B	C	D	E		
B	C	D	E			A
C	D	E			A	B
D	E			A	B	C
E			A	B	C	D
		A	B	C	D	E
	A	B	C	D	E	

The above plan could be augmented with treatments 0, 1, 2, 3, 4, 5, and 6 such that two of the three transversals required to obtain balance are selected. These treatments would form a partially balanced incomplete block design arrangement within rows and within columns; the resulting design,

A	B	C	D	E	5	6
B	C	D	E	6	0	A
C	D	E	0	1	A	B
D	E	1	2	A	B	C
E	2	3	A	B	C	D
3	4	A	B	C	D	E
5	A	B	C	D	E	4

,

is of the $0{:}S_pS_p$ type.

8. Other Augmented Plans

Suppose that the basic design is the $n_1 \times n_1$ part of plans similar to $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ from which p transversals, $p = 1,2,\ldots n_1-2$, have been projected and that the basic design is either of the T:TT type or the P:PP type. Denote the elements or treatments in the basic design as the original treatments. If the pn_1 empty cells in $C_3(n_1 + n_2)$ type plans are filled with pn_1 additional entries each occurring once and if the p = k projected transversals form a balanced incomplete block design arrangement, the resulting plan is called an augmented T:TT type of plan. An example of an augmented T:TT plan with pn_1 = 3(21) additional treatments is obtained from $C_3(10)$ as follows:

0	a	b	3	c	5	6
d	e	4	f	6	0	1
g	5	h	0	1	2	i
6	j	1	2	3	k	ℓ
m	2	3	4	n	o	0
3	4	5	p	q	1	r
5	6	s	t	2	u	4

The 21 augmented treatments are denoted by the letters a, b, c, d,...,t, and u, and each occur once. The treatments denoted by 0, 1, 2, 3, 4, 5, and 6 occur four times each.

Another form of augmentation for a T:TT plan similar to the upper left-hand partitioning of $C_3(10)$ is to include pn_1/r_a = 7 treatments each occurring r_a = 3 times each. If the augmented treatments are included in such a manner as to form a T:TT plan by themselves and if these 7 treatments are denoted by the letters a, b, c, d, e, f, and g, a plan of the following type results:

0	b	c	3	e	5	6
c	d	4	f	6	0	1
e	5	g	0	1	2	d
6	a	1	2	3	e	f
b	2	3	4	f	g	0
3	4	5	g	a	1	c
5	6	a	b	2	d	4

considering all 14 treatments, the above 7 x 7 plan is of the O:PP type and has a two-associate class partially balanced incomplete block design arrangement within rows and within columns.

From the 7 x 7 part of $C_1(10)$ several augmented plans may be obtained. Some of these are:

Transversals projected(no.)	Augmented treat.(no.)	Total no. of treatments	Replicates for augmented treat.(no.)	Replicates for orig. treat.(no.)
1,2,3,4,5,6	1	8	7,14,21,28,35,42	6,5,4,3,2,1
2,4,6	2	9	7,14,21	5,3,1
3,6	3	10	7,14	4,1
4	4	11	7	3
5	5	12	7	2
6	6	13	7	1
1,2,3,4,5,6	7	14	1,2,3,4,5,6	6,5,4,3,2,1
2,4,6	14	21	1,2,3	5,3,1
3,6	21	28	1,2	4,1
4	28	35	1	3
5	35	42	1	2

In addition to the above plans with two sets of replication, other augmented plans may be constructed for three, four,... sets of replication for the sets of treatments. For example, suppose that eight additional treatments are to be used.

One treatment could be replicated $n_1 = 7$ times on one transversal; the other seven additional treatments could be replicated two times each and inserted on two transversals and in a partially balanced incomplete block design arrangement within rows and within columns; the original $n_1 = 7$ treatments could be included four times each in the T:TT plan from $C_3(10)$. Thus, there is supplemented balance between the original seven treatments and the treatment included seven times; there is supplemented partial balance between the additional treatment included seven times and the remaining augmented treatments which are included twice; the augmented treatment occurring $n_1 = 7$ times is in supplemented partial balance to the remaining 14 treatments in the plan. This design is of the $O{:}S_pS_p$ type with three different numbers of replicates for the $1 + 7 + 7 = 15$ treatments.

The above illustrates the great diversity of augmented designs available for experimentation. Other augmented designs have been discussed by Federer [1956, 1960, 1961].

9. Plans for the Design of Successive Experiments

If two sets of treatments in two separate experiments are conducted successively or simultaneously on the same set of experimental units, it is desirable to have plans in which the rows, columns, and each set of treatments are as nearly orthogonal as possible. Orthogonal experiments are more efficient than nonorthogonal experiments in the sense that smaller variances of treatment differences are obtained. Plans with various properties among the two sets of treatments and the rows and columns have been constructed by various authors (see Hoblyn et al.[1954], Pearce [1960, 1963], Clarke [1963], Freeman [1964], and Hedayat et al.[1970]. Designs of the O:OT:TOO and O:OO:SSS types have been constructed, where the first three letters and the first colon have the meaning used previously and where the three letters to the right of the second colon refer to the properties of the second set of treatments with respect to rows, columns, and the first set of treatments.

We may use plans of the form of $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$ to construct 3×7, 4×7, 5×11, $6 \times 11, \ldots, k \times n_1$, and $(n_1-k) \times n_1$ Youden square plans of the O:OT:OTT type for successive experiments. For example, if the first seven rows and last three columns of $C_3(10)$ are superimposed on the corresponding rows and columns of $C_4(10)$, a 7-row $\times$ 3-column design of the O:TO:TOT type results. Similarly superimposing the last three rows and the first seven columns of $C_3(10)$ on the corresponding ones of $C_4(10)$ produces a 3-row $\times$ 7-column design of the O:OT:OTT type. The use of the remaining four transversals in $C_3(10)$ and $C_4(10)$ projected into rows and columns and superimposing one set on the other results in a 7-row $\times$ 4-column plan of the O:TO:TOT type and a 3-row $\times$ 7-column plan of the O:OT:OTT type.

Using the same procedure described above, we may construct 5-row $\times$ 11-column and 6-row $\times$ 11-column plans of the O:OT:OTT type and 11-row $\times$ 5-column and 11-row $\times$ 6-column plans of the O:TO:TOT type. In general, whenever a Youden square plan can be formed from the projection of k transversals into the last k rows of $C_3(n_1 + n_2)$ and $C_4(n_1 + n_2)$, superimposition of the corresponding parts produces a k-row $\times$ n_1-column O:OT:OTT plan.

An alternative procedure for constructing O:OT:OTT plans is to obtain a pair of orthogonal latin squares such that one square is a cyclical permutation of the rows of the other square. This is possible for all odd n_1 (see Hedayat and Federer [1969]). Then, divide the n_1 rows into two sets of rows for both squares such that n_1-k rows form a Youden square and also that the remaining k rows form a Youden square. The same permutation of rows is used for both squares from the $O(n_1,2)$ set. Using $L_1(7)$ and the latin square of order 7 from $C_1(10)$ to form the $O(7,2)$ set, the following plan illustrates the procedure:

A1	B2	C3	D4	E5	F6	G7	Youden square plans
B3	C4	D5	E6	F7	G1	A2	for $v = 7 = n_1$, $k = 4$
E2	F3	G4	A5	B6	C7	D1	$= r$, and $\lambda = 2$.
G6	A7	B1	C2	D3	E4	F5	
C5	D6	E7	F1	G2	A3	B4	Youden square plans
D7	E1	F2	G3	A4	B5	C6	for $v = 7 = n_1$, $k = 3$
F4	G5	A6	B7	C1	D2	E3	$= r$, and $\lambda = 1$.

The above procedure produces plans of the O:OT:OTT type.

If three sets of successive experiments are to be conducted with three sets of treatments, an $O(n_1,3)$ set may be used to construct plans, e.g., of the O:OT:OTT:OTTT type where the set of letters OTTT refers to the relationship among the third set of treatments to the rows, the columns, the first set of treatments, and the second set of treatments. Likewise, $O(n_1,t)$ sets may be used to construct plans for t successive experiments on the same material.

Numerous additional plans of the augmented type may be constructed. The following example illustrates one such type wherein an augmented O:TT type plan is superimposed on a latin square of order 7:

A0	Ba	Cb	D3	Ec	F5	G6
Bd	Ce	D4	Ef	F6	G0	A1
Cg	D5	Eh	F0	G1	A2	Bi
D6	Ej	F1	G2	A3	Bk	Cℓ
Em	F2	G3	A4	Bn	Co	D0
F3	G4	A5	Bp	Cq	D1	Er
G5	A6	Bs	Ct	D2	Eu	F4

The elements A, B, C, D, E, F, and G replace elements 0, 1, 2, 3, 4, 5, and 6 in $L_1(7)$ and the empty spaces in $C_3(10)$ are replaced by the letters a, b, c,...,t, and u.

In addition to the above types of designs, the $n_1 \times n_1$ parts of plans like $L_1(n_1 + n_2)$ and $L_2(n_1 + n_2)$ formed by the sum composition method may be superimposed on each other to form an n_1-row $\times$ n_1-column design with $n_1 + n_2$ treatments in the first experiment and $n_1 + n_2$ treatments in the second experiment.

Other permutations of plans are possible and numerous but the above suffices to demonstrate the usefulness of the sum composition method of constructing $O(n_1 + n_2, 2)$ sets and of common-parallel transversals in latin squares for constructing several classes of experimental designs. These designs are useful for experimental material requiring control of heterogeneity from two sources of variation.

10. Summary

An experiment should be designed to satisfy the experimental considerations and objectives of the experiment. The experiment should not be designed to fit into known experimental designs *if* this results in a change in the desired findings. Consequently, new experimental designs and concepts need to be developed to satisfy new experimental requirements which continue to arise as a result of new investigations and research interests.

Several new types of experimental designs and some new methods of constructing known designs are considered in this paper. The ideas of parallel transversals in a latin square of order n and of common-parallel transversals in a pair of orthogonal latin squares of order n proved useful in constructing new as well as some previously known designs. The ideas and the procedure involved in the sum composition method of constructing a pair of orthogonal latin squares of order n proved very useful in the present work. The sum composition method is a new procedure for constructing a pair of orthogonal latin squares and was developed by A. Hedayat. It derives its name from the fact that use is made of pairs of orthogonal latin squares of order n_1 and n_2, such that $n_1 + n_2 = n$ and $n_1 \geq 2n_2$, to produce the larger pair of orthogonal latin squares of order n. The method makes use of the projection of parallel and common-parallel transversals into the last n_2 rows and columns of the square of order n.

It is shown how to construct Youden square and balanced incomplete block designs using the sum composition method. Plans of partially balanced latin rectangle designs, T:TT type designs, supplemented balanced and partially balanced designs, augmented designs, and designs for successive experiments conducted on the same experimental units were constructed using the ideas and procedures in the sum composition method. Most of the plans constructed will have a relatively simple statistical analysis. The analyses are not given in the present paper.

11. Literature Cited

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